University of Maryland, College Park
Department of Aerospace Engineering

ADA085649

Office of Naval Research

Contract N00014-76-C-0872

Report No. 80-1

An Extended Field Method Solution For A

Pulse Loaded Thin Plate

by

Bruce K. Donaldson

and

N. B. Starkey

80 6 10 018

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AE-80-1	2. GOVT ACCESSION NO. AD-A085 649	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) An Extended Field Method Solution for a Pulse Loaded Thin Plate		5. TYPE OF REPORT & PERIOD COVERED Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Bruce K. Donaldson N. B. Starkey		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0872
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Maryland College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Structural Mechanics Program Arlington, VA 22217		12. REPORT DATE Apr 1980
		13. NUMBER OF PAGES 11
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

extended field method
pulse loading
plate
series solution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with a new extension of a weighted residual method of analysis called the extended field method. The extended field method is applied for the first time to the problem of the transient vibration of a uniformly thin elastic plate. Numerical results have been obtained which validate the analysis procedure and show better solution convergence that is obtainable by standard methods of analysis for the same number of degrees of freedom. Further studies are necessary to obtain still better convergence.
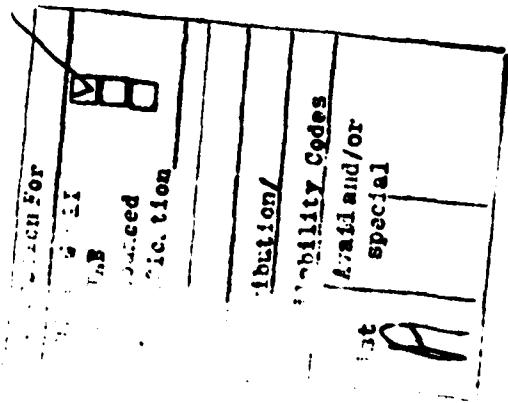
401860

ABSTRACT

This paper deals with a new extension of a weighted residual method of analysis called the extended field method. The extended field method is applied for the first time to the problem of the transient vibration of a uniformly thin elastic plate. Numerical results have been obtained which validate the analysis procedure and show better solution convergence than is obtainable by standard methods of analysis for the same number of degrees of freedom. Further studies are necessary to obtain still better convergence.

NOMENCLATURE

a	extended plate length in x-coordinate direction
b	extended plate length in y-coordinate direction
c,$\bar{c}$	extended plate offsets in x-coordinate direction
d	plate thickness
e,$\bar{e}$	extended plate offsets in y-coordinate direction
$h_{j,n,k}$	n,kth Fourier series coefficient of the extended plate side j arbitrary lateral displacements, where j = 1,2,3,4
$k_{j,n,k}$	n,kth Fourier series coefficient of the extended plate side j aribtrary normal slope where j = 1,2,3,4
k,m,n,p,q	summation indices
p(x,y;t)	pressure loading over the plate area
$P_{m,n}$	double Fourier series coefficient of the pressure loading

1

p_0	spatially uniform pressure amplitude
t	time
t_0	time duration of pressure pulse loading
$w(x,y;t)$	plate lateral deflection function
$w_5(x,y;t)$	particular solution component of $w(x,y;t)$
x,y	plate Cartesian coordinates
D	plate stiffness factor, $Ed^3/(12(1-\mu^2))$
E	Young's modulus
$K,M,N,\bar{N}$	maximum values for the indices
σ_i	1 if i is odd; 0 otherwise
$Q_{i,j}, Q_{1,i,j}, Q_{2,i,j}, R_{i,j}$	constants of integration for complementary solution to the nonhomogeneous equation (10)
$W_{j,i}(x,y)$	plate lateral deflection amplitude; the subscript j refers to a specific set of boundary conditions, and the subscript i ranges to N
$\alpha_{m,n}$	$(D/\rho d)^{\frac{1}{2}}[(m\pi/a)^2 + (n\pi/b)^2]$
$\beta_{m,n}$	$(16p_0\sigma_m\sigma_n)/(\pi^2 mn\rho d)$
μ	Poisson's ratio
ρ	mass density
ω_k	$k\pi/T$, where $k = 1,2,\ldots,N$

INTRODUCTION

The purpose of this paper is to present an extension of an approximate method of analysis called the extended field method (XFM). In brief, the XFM employs finite series which are term-by-term solutions to the governing differential equation(s). Then the device of an offset auxiliary boundary, which is the boundary of the "extended field", is coupled with the Galerkin error minimization technique in order to approximate the given boundary conditions [1]. The necessity for term-by-term series solutions to the governing differential equation limits the XFM to simple geometries. Despite this lack of versatility, the XFM is noteworthy because of its demonstrated superior solution convergence.

In its previous development the XFM has been applied to the problem of uniformly thin, elastic, polygonal plates undergoing forced harmonic vibration, and small combinations of plates and open section beams undergoing forced harmonic vibration. For the purposes of discussing the previous displacement amplitude numerical results, single plate analyses are divided into two groups. The first group consists of those analyses where the polygonal plates possess one or more free edges. This is a difficult group for the XFM because the free edge boundary conditions require three differentiations of the XFM finite series solutions, and there is a consequent lessening of the convergence rate. For this group of plates, as many as 150 degrees of freedom may be necessary to achieve a completely convincing four significant digits of convergence. Even

so, this is more in the way of solution convergence than can be obtained with many more degrees of freedom using the more popular methods of analysis such as the finite element, finite difference, and Rayleigh-Ritz methods [2].

The second group of analyses concerns polygonal plates without free edge boundary conditions. For this group of analyses, the XFM generally attains at least six significant figures of convergence with as little as one hundred degrees of freedom. In particularly favorable analyses, seven significant figures of convergence can be attained with that number of degrees of freedom. Thus for the problem of linear, forced harmonically vibrating plates, the XFM can provide benchmark solutions. However, forced vibration problems are a relatively simple set of problems.

This paper presents XFM numerical results for a somewhat more complicated problem, that of the linear response of a uniformly thin plate subjected to an arbitrary lateral pressure pulse. In this problem the time variable, t, cannot be deleted from the analysis, and initial conditions as well as boundary conditions must be approximated. Nevertheless, just as there is no need for any sort of geometric grid in order to describe spatial variations, the XFM does not use any sort of step-by-step integration process to evaluate the time varying response. Furthermore, there is no difference in style between this XFM analysis and the simpler forced harmonic analysis which forms a portion of the present analysis. The succeeding sections will present the analysis procedure, the numerical results, and the conclusions that may be drawn.

THE ANALYSIS PROCEDURE IN BRIEF

In Cartesian coordinates the governing differential equation for a thin, uniform, isotropic, homogeneous, linearly elastic plate subjected to an arbitrary time varying pressure loading is

$$D \nabla^4 w(x,y;t) + (\rho d)w_{,tt}(x,y;t) = p(x,y;t) \tag{1}$$

where a comma indicates partial differentiation. The solution to this equation is obtained in several parts without regard to the shape of the actual plate boundary or the nature of the actual boundary supports. First consider the homogeneous form of Eq. (1). The homogeneous form can be solved by first separating the spatial and temporal variables so that

$$w(x,y;t) = W(x,y)\sin\omega t \tag{2}$$

Then the homogeneous equation becomes

$$\nabla^4 W(x,y) - (\rho\omega^2 d/D)W(x,y) = 0 \tag{3}$$

This homogeneous equation is the same as the homogeneous equation associated with forced harmonic vibration. Thus the same four Lévy series solutions used in the harmonic case [1] may also be used here. However there is the difference that in this nonharmonic case, the frequency of forced vibration, ω, has no meaning. In order to determine appropriate values for ω, define a time period of interest $(0,T)$ over which the

3

solution will be sought. In engineering terms, this time period is simply defined. It would usually be a multiple of the plate's first natural period or a multiple of some characteristic time interval associated with the applied loading. Therefore, with the pulse beginning at $t = 0$, write

$$\omega = \omega_k = \frac{k\pi}{T}, \quad k = 1,2,\ldots,K \tag{4}$$

and there will be four Lévy series solutions for each value of k. Ignoring the special case where the boundary conditions involve adjacent free edges, which is dealt with in Ref. 2, the solution to Eq. (3) is

$$w(x,y,t) = \sum_{j=1}^{4} \sum_{k=1}^{K} W_{j,k}(x,y)\sin\omega_k t \tag{5}$$

where

$$W_{1,k}(x,y) = \sum_{n=1}^{N} J_{n,k}\{h_{1,n,k}[s_{n,k}\cos(h)s_{n,k}a\,\sinh r_{n,k}(a-x)$$

$$-r_{n,k}\cosh r_{n,k}a\,\sin(h)s_{n,k}(a-x)]$$

$$+ k_{1,n,k}[\sin(h)\,s_{n,k}a\,\sinh r_{n,k}(a-x)$$

$$- \sinh r_{n,k}a\,\sin(h)s_{n,k}(a-x)]\}\sin\frac{n\pi y}{b}$$

$$J_{n,k} = [s_{n,k}\sinh r_{n,k}a\cos(h)s_{n,k}a - r_{n,k}\cosh r_{n,k}a\,\sin(h)s_{n,k}a]^{-1}$$

$$r_{n,k} = [(n\pi/b^2) + \omega_k(\rho d/D)^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$s_{n,k} = [(n\pi/b^2) - \omega_k(\rho d/D)^{\frac{1}{2}}]^{\frac{1}{2}} \quad \text{for } n > \hat{n}_k$$

$$= [\omega_k(\rho d/D)^{\frac{1}{2}} - (n\pi/b)^2]^{\frac{1}{2}} \quad \text{for } n < \hat{n}_k$$

$$\hat{n}_k = (b/\pi)(\rho\omega_k^2 d/D)^{\frac{1}{4}}$$

$$\sin(h)s_{n,k}x = \sin s_{n,k}x \qquad \text{if } n < \hat{n}_k$$

$$\sinh s_{n,k}x \qquad \text{if } n > \hat{n}_k$$

$$\cos(h)s_{n,k}x = \cos s_{n,k}x \qquad \text{if } n < \hat{n}_k$$

$$\cosh s_{n,k}x \qquad \text{if } n > \hat{n}_k \tag{6}$$

As a brief explanation, for example, the Lévy series $W_{1,k}$ can be thought of as representing the solution for a rectangular plate of dimensions a by b with simple supports at sides $y = 0,b$ and $x = a$, but arbitrary lateral displacements and normal slopes at side one $(x = 0)$ that are described by the respective equations

$$W(0,y) = \sum_{n=1}^{N} \sum_{k=1}^{K} h_{1,n,k}\sin\frac{n\pi y}{b}$$

4

$$W_{,x}(0,y) = \sum_{n=1}^{N} \sum_{k=1}^{K} k_{1,n,k} \sin \frac{n\pi y}{b} \qquad (7)$$

This rectangular plate is called the extended plate. Later in the analysis, the rectangular extended plate boundaries will enclose the actual plate boundaries. The third Lévy series, $W_{3,n,k}$, which corresponds to arbitrary boundary conditions on the extended plate edge $x = a$, is like the first series, but with x replacing $a - x$ and $-k_{3,n,k}$ replacing $+k_{1,n,k}$. The second series, corresponding to arbitrary conditions at $y = 0$ can be obtained from the first by interchanging a,x and b,y. The fourth solution can be obtained from the second as the third is obtained from the first.

To obtain the particular solution for the non-homogeneous equation (1) write the forcing pressure in double Fourier series form as

$$p(x,y;t) = \sum_{m=1}^{M} \sum_{n=1}^{M} p_{m,n}(t)\sin(m\pi x/a)\sin(n\pi y/b) \qquad (8)$$

where the index M is large enough to ensure satisfactory convergence. Then the particular solution may be written in Navier series form as

$$w_5(x,y,t) = \sum_{m=1}^{M} \sum_{n=1}^{M} Z_{m,n}(t)\sin(m\pi x/a)\sin(n\pi y/b) \qquad (9)$$

Substitution of Eq. (9) into Eq. (1), and noting the linear independence of the sine functions, leads to

$$\ddot{Z}_{m,n}(t) + \alpha_{m,n}^2 Z_{m,n}(t) = p_{m,n}(t)/\rho d \qquad (10)$$

where

$$\alpha_{m,n}^2 = (D/\rho d)][(m\pi/a)^2 + (n\pi/b)^2]^2.$$

The solution of these M^2 elementary ordinary differential equations completes the solution for $w_5(x,y,t)$, and hence for $w(x,y,t)$ which, again, is the right hand side of Eq. (5) plus $w_5(x,y;t)$.

In order to illustrate the above, consider a fully clamped rectangular plate of dimensions $a-c-\bar{c}$ by $b-e-\bar{e}$ where $c,e,\bar{c},\bar{e}$ are the offsets between the actual plate rectangular boundary and the extended rectangular plate boundary. Let the applied loading, which is extended to the boundaries of the extended plate as well, be

$$p(x,y,t) = \begin{cases} p_0 \sin(\pi t/t_0) & 0 \le t \le t_0 \\ 0 & \text{otherwise} \end{cases} \qquad (11)$$

This choice of plate and loading for an example problem is neither favorable or unfavorable for the analysis procedure. The four clamped edges require a larger number of degrees of freedom than is necessary in many cases, and the double Fourier series representation of $p(x,y;t)$ is slow to converge. On the other hand, the absence of one or more free edges improves the convergence rate [2]. Substitution of Eq. (11)

into Eq. (8) yields

$$p_{m,n}(t) = (16p_0\sigma_m\sigma_n/\pi^2 mn)\sin(\pi t/t_0) \qquad (12)$$

where σ_m is plus one if m is an odd integer, and zero if even. Then the solution to Eq. (10) is easily found to be such that, for $0 \leq t \leq t_0$

$$w_5(x,y,t) = \sum_{m=1}^{M} \sum_{n=1}^{M} \{Q_{m,n}\sin\alpha_{m,n}t + R_{m,n}\cos\alpha_{m,n}t$$

$$+ \frac{\beta_{m,n}}{\alpha_{m,n}^2-(\pi/t_0)^2} \sin(\pi t/t_0)\}\sin(m\pi x/a)\sin(n\pi y/b) \qquad (13)$$

where $\beta_{m,n} = 16p_0\sigma_m\sigma_n/\pi^2 mn\rho d$ and $Q_{m,n}$ and $R_{m,n}$ are constants of integration. The time period after $t = t_0$ can be dealt with by standard super-position procedure involving two continuous sinusoidal loadings with a half period phase shift.

Now it is necessary to satisfy the initial conditions and the actual plate boundary conditions. The initial conditions will be satisfied first. Let these conditions be those of zero initial displacement and zero initial velocity. Zero initial displacement quickly leads to $R_{m,n} = 0$. To satisfy zero initial velocity, K is set equal to N, and $Q_{m,n}$ is split into two parts. $Q_{1,m,n}$ is used to cancel the w_5 portion of the initial velocity equation, and $Q_{2,m,n}$ cancels the W_1 through W_4 portion. In other words, due to the linear independence of the sine functions

$$Q_{1,m,n} = \frac{-\pi\beta_{m,n}/\alpha_{m,n}t_0}{\alpha_{m,n}^2 - (\pi/t_0)^2} \qquad m,n = 1,2,\ldots,M \qquad (14)$$

and $Q_{2,m,n}$ satisfies the equation

$$\sum_{m=1}^{N} \sum_{n=1}^{N} Q_{2,m,m}\, \alpha_{m,n}\, \sin(m\pi x/a)\sin(n\pi y/b) \qquad (15)$$

$$+ \sum_{k=1}^{N} \frac{k\pi}{T} [W_{1,k} + \ldots + W_{4,k}] = 0 \qquad (16)$$

where $Q_{2,m,n}$ is zero for m,n greater than N. (Note $M \gg N$.) Eq. (15) is written as an equality, but it is actually only an approximation. The error of this approximation may be minimized by use of the Galerkin technique with weighting functions $\sin(p\pi x/a)$ and $\sin(q\pi y/b)$ on the intervals $(0,a)$ and $(0,b)$. For the sake of brevity, the algebraic solution for $Q_{2,m,n}$ will be omitted here.

The satisfaction of the eight boundary conditions is even more straight forward. For example, the boundary condition $w(c,y;t) = 0$ is approximated by substitution of the expression obtained above for

6

w(x,y;t). Then the error of that approximation to the boundary condition w(c,y,t) = 0 is minimized by the Galerkin technique along the length of the actual edge x = c by use of the weighting function sin(pπy/b) on the interval (0,b) and minimized again over the time interval (0,T) with the weighting function sin(qπt/T), where p,q = 1,2,...,N. The result is N^2 linear, non-homogeneous algebraic equations in the $8N^2$ unknown extended plate edge coefficients $h_{1,1,1}$ through $k_{4,N,N}$. The other seven boundary condition equations supply the remaining $7N^2$ equations. Ref. 3 details the above derivation and lists all necessary results, specifically, the unabridged forms for the eight boundary condition equations whose solution explicitly defines w(x,y;t).

NUMERICAL RESULTS

The fully clamped rectangular plate subjected to the half-period sine pulse discussed above was programmed for the UNIVAC 1140 digital computer. The selected input for the plate geometry and material properties was a = 20 inches, b = 16 inches, d = 1 inch, $c = e = \bar{c} = \bar{e} = 0.0$, E = 10,500,000 psi, μ = 0.30, and ρ = 0.000254 lb-sec^2/in^4. The selected parameters for the pressure loading were a uniform spatial amplitude of $p_0 = 1$ psi, and a half sine pulse duration of $t_0 = 0.010873$ sec. The latter number, besides having too many digits for an essentially arbitrary choice, was based on an originally mistaken estimate of the plate's first natural period. A correct estimate is 0.00087 secs. [4]. Thus the selected pulse duration turned out to be approximately twelve and one-half times the plate's first natural period. Thus it was reasonable to set the time period of interest, $T = t_0$. In order to well define the response in the interval (0,T), it was decided to calculate the displacement response at 101 equally spaced points, including the end points.

The primary purpose for gathering the numerical results presented here was to investigate the degree of solution convergence obtainable by the XFM analysis, and thus learn if the same high degree of convergence that characterized forced harmonic vibration XFM solutions could also be obtained for pulse loadings. In order to study the XFM solution convergence, three indices were varied. The first of these was N, the maximum number of terms in each solution series. The importance of N is that the number of degrees of freedom that describe the plate displacements is $8N^2$. This index, as expected, was the critical index.

The second index was $\bar{N}$, another maximum summation index originally equal to N, but by rearranging certain orders of summation, could be freed from being tied to the small value that N must normally be. The results in Table 1 show that a value of $\bar{N} = 25$ is quite sufficient. The third index that was varied is M, which determines the number of series terms used to describe the input pressure. As Table 1 suggests, M, which is an odd number because even indexed input terms are zero, is less

7

influential than N. In the two instances checked, only the sixth significant digit was affected when M was increased from 49 to 99, and in the one instance checked, only the seventh significant digit was affected when M was increased from 99 to 149. Thus only N needs further consideration when discussing convergence at this time.

Before studying the effect of N on convergence it is worth noting that the digital computer program that supplied the results presented in Table 1 was verified by comparison to a finite element, direct stepwise integration solution. Specifically, an XFM solution at $N = 8$,

$\overline{N} = 49$, $M = 99$ for the same plate, but for a change in thickness to 0.1 inch, was within one half percent of the corresponding NASTRAN solution using eighty QUAD2 elements and 364 degrees of freedom over one quarter of the doubly symmetric, actual plate. This is a good place to mention that although the XFM solution could also have been greatly simplified by use of the double symmetry of the plate geometry and loading, no such simplification was made in order to present results that equally well characterize a loading with any spatial distribution.

The digital listing of the time response curve for a particular geometric point on the plate represents a lot of different numbers. Response curves for different maximum indices could have been compared on the basis, for example, of an averaged sum of the absolute values of the displacement differences, or some similar scheme that compares the entire curves on a weighted or unweighted basis. However, from an engineering point of view, the maximum displacement is usually the most interesting displacement measurement. Of course, the true maximum lies between data points. Table 1 presents the values at that data point (time instant) closest to where the true maximum occurs. In all cases studied it was the same time instant, number 56, which is equivalent to an elapsed time of 0.00598015 sec. Returning now to the convergence pattern dependent upon N. It suffices to look at the data for which

$M = 99$ and $\overline{N} = 49$. The plate center displacements for $N = 3, 4$, and 5 have only converged to the third significant digit. This degree of convergence is typical of the convergence at other time points, with very little deviation in the convergence rate from one time point to another. This situation is to be expected since the solution error is essentially the same at all instants of time. That is, the error does not increase with time as it does with a step-by-step integration scheme. The same background comments apply for the values of N equal to six and seven. Here it is almost possible to claim a fourth digit of convergence because the Cauchy differences are decreasing rapidly. Specifically, the

Cauchy differences multiplied by 10^9 for the pairs of values of N equal to (5,4), (6,5), and (7,6) are 67, 50, and 5 respectively. However, it is not possible to guarantee that the above trend will continue. Therefore only convergence to a third digit will be claimed, or in other terms, the maximum center displacement appears to be $0.1304 \pm .0004 \times 10^{-3}$ inches.

8

| N | Degrees of Freedom | $\bar{N}$ | DISPLACEMENTS (INCHES X 1,000) | | |
			M = 49	M = 99	M = 149
3	72	25	.1303 7242	.1303 7124	
		49		.1303 7124	
4	128	25	.1302 9999	.1302 9881	.1302 9869
		49		.1302 9881	
		75		.1302 9881	
5	200	25		.1303 6552	
		49		.1303 6552	
6	288	49		.1304 1552	
7	392	49		.1304 2131	

TABLE 1

Plate Center Displacement for Various Maximum Index Values at Time
Equal to 0.00598015 sec Which Corresponds to 0.55 of the
Pulse Duration, the Time Point Closest to the
Time of Maximum Displacement

CONCLUSIONS AND FUTURE WORK

From the comparison to the NASTRAN result it is clear that the XFM does provide correct solutions for the pulse loaded thin plate problem. The three significant digit convergence at two hundred degrees of freedom is not spectacular, but it is still distinctly better than can be obtained from a standard finite element, finite difference, or Rayleigh-Ritz analysis. It is also clear that solutions for increased values of N are required to more fully explore the XFM convergence rate. Other parametric studies are also needed. Variations on the relations between the pulse duration, the first natural period, and the period of interest need investigation. Reasonable variations on the offset distances and pulse shape are not expected to have any effect on solution convergence but this also should be verified.

REFERENCES

1. Donaldson, B.K., "A New Approach to the Forced Vibration of Thin Plates," Journal of Sound and Vibration 30(4), 397-417, 1973.

2. Chander, S., Donaldson, B.K., and Negm, H.M., "Improved Extended Field Method Numerical Results," Journal of Sound and Vibration 66(1), 39-51, 1979.

3. Starkey, N.B., "The Extended Field Method Applied to the Non-Harmonic Loading Condition," M.S. Thesis, University of Maryland Dept. of Aerospace Engineering, 1978.

4. Leissa, A.W., "Vibration of Plates," NASA SP-160, 1969.